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THE

SHREWSBURY TRIGONOMETRY.

A STEP TO THE STUDY OF A MORE COMPLETE TREATISE.

FOR BEGINNERS, AND JUNIOR CLASSES IN PUBLIC SCHOOLS.

BY THE

REV. J. C. P. ALDOUS,

LATE SCHOLAR, OF KEBLE COLLEGE,
AND MATHEMATICAL MASTER, SHREWSBURY SCHOOL.

CAMBRIDGE:
DEIGHTON, BELL, AND CO.
LONDON: G. BELL AND SONS.
1874.

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PREFACE

THIS book has been written for the use of boys, and also as a help to those who wish to read for themselves the elementary and practical parts of Trigonometry.

In my experience of teaching, I have found that the first two terms spent by a boy of ordinary power on Todhunter's larger Trigonometry have been all but wasted.

He has been there brought for the first time face to face with bookwork, most exactly and logically arranged. This must be learned of itself, and no explanation or illustration is added.

There is also a deficiency of that which formed the bulk of other subjects, namely of examples in working which he might gain an insight into the principles of the bookwork and the use to be made of it.

It has been my object to provide an introduction to Mr Todhunter's excellent Manual, and to write a simple book, which shall contain enough matter for a term's reading, and examples sufficiently easy for a boy to work out unaided, as evidence to himself that he understands what he has read.

It gives me great pleasure to think that this little book bears a tribute to the memory of the late Canon Temple Chevallier, of Durham, who read it and gave me many very valuable suggestions.

I have been also very much assisted in the idea and progress of the work by my Tutor and Friend, Mr J. F. Moulton, of Christ's College.

J. C. P. A.

Jan. 1*st*, 1874.

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CHAPTER I.

ANGLES AND THE MEASUREMENT OF ANGLES.

1

A plane rectilineal angle is the inclination of two straight lines to one another, which meet together, but are not in the same straight line.

This is one of Euclid's Definitions, which describes the nature of the angles treated of in Plane Trigonometry.

The inclination of one line to another, that is, the angle between the two lines, may be increased or diminished; it may also be doubled or halved, or any multiple may be taken of it. Also one angle may be said to be greater than another or less.

Hence to distinguish and describe an angle it is natural to take a measure, for we see that angles differ in magnitude, and measurement can be applied to them as well as to lengths and other quantities.

Now a measure or standard must be of the same kind as the thing to be measured, and must be of a constant size.

Acres cannot be measured by pints, nor can we be sure that we have 30 yards of cloth when that quantity

has been measured to us unless all yard measures are of the same length.

2 There is one well-known angle by which angles may be measured.

DEF. When one straight line standing on another straight line makes the adjacent angles equal to one another, each of these angles is called **a Right Angle.**

Now all right angles are equal to one another. So that any angle may be measured by them.

For example, $\frac{3}{4}$ of a right angle or $2\frac{3}{4}$ of a right angle denote certain angles which we have expressed in terms of this measure or standard angle.

We may refer to the Mariner's Compass as a use of the right angle as a measure; for when a pilot wishes the course of a ship altered, he speaks of turning it through two or three points; the right angle being divided into eight points this denotes that it is turned through $\frac{1}{4}$ or $\frac{3}{8}$ of a right angle.

And as the ship might be turned round and round we might imagine it as turned through any number of points. Though we cannot draw an angle greater than two right angles, we see in this way that there is really **no limit to its size.**

3 It has been shewn that an angle may consist of any number of right angles or parts of a right angle, that is, an angle will be known if we know what fraction or multiple it is of a right angle.

But it would be inconvenient to measure an angle only by its relation to a right angle, just as it would be

hard to tell the time of day by the fraction of the year that has passed.

So the right angle is divided into different denominations, as a year is divided into months, days, hours, minutes and seconds.

The tables in use for angles are two; one in England, and one in France.

English Measure.	French Measure.
60″ sec. = 1′ minute	100‴ sec. = 1‵ minute
60′ min. = 1° degree	100‵ min. = 1ᵍ grade
90° deg. = a right angle	100ᵍ grades = a right angle

Both systems refer to a right angle, dividing it either into 100ᵍ or 90°; so that, **10 grades make 9 degrees.**

Hence, to reduce degrees to grades,

multiply by 10 and divide by 9,

while to reduce grades to degrees,

multiply by 9 and divide by 10.

Remember that there are **more grades than degrees** in a right angle, or any angle, and it will be seen whether the multiplier is 10 or 9, in each case.

To reduce parts of a degree to grades. Bring the min. and sec. to a fraction of a degree, thus 13′ 20″ $= \frac{800}{3600}$ of a degree. This fraction must be multiplied by 10, and divided by 9, $= \frac{80}{324}$ grades $= 24‵.69‴$, nearly.

The parts of a grade are easily expressed in a decimal fraction, *e.g.* $1ᵍ, 3‵.21‴ = 1{\cdot}0321$ grades.

To reduce this to the decimal part of a degree. Multiply by 9 and divide by 10 *by moving the decimal point one place* = ·92889 degrees = 55′ 44″ nearly.

These methods are merely applications of Arithmetical Reduction, and a few examples will suffice for practice in applying them.

EXERCISES IN REDUCTION.

Reduce to degrees, minutes and seconds,

1. $\frac{1}{3}$ of a right angle; $\frac{1}{25}$; $\frac{1}{30}$; $\frac{1}{4}$; ·301 of a right angle.

2. 23^g; 25^g; 48^g; $35^g.21'$; $18^g.21'.30''$; 335^g.

3. $\frac{1}{18}$ of a right angle; $\frac{1}{225}$; $\frac{21}{436}$; $\frac{51}{282}$; $\frac{18}{311}$; $\frac{426}{13}$ of a right angle.

4. $225^g.18'.1''$; $23^g.14'.12''$; $314^g.1'.2''$.

5. The angle of an octagon; a dodecagon.

6. The angle contained by the hands of a clock at $5^h.20^m$; $6^h.32'$; 5 min. to 12.

Reduce to grades, minutes and seconds,

1. $\frac{1}{30}$; $\frac{1}{6}$; $\frac{1}{7}$; $\frac{1}{21}$; $\frac{1}{31}$; ·301030 of a right angle.

2. $16°.13'.41''$; $21°$; $52°$; $83°$; $45°$; $13°.14'.8''$.

3. $\frac{2}{3}$; $\frac{5}{21}$; $\frac{14}{31}$; $\frac{8}{13}$; $\frac{83}{121}$; $\frac{43}{13}$ of a right angle.

4. $48°.31'$; $16°.36'$; $14°.32'.23''$; $63°.8'.2''$.

5. The angles of an isosceles triangle, each of the angles at the base being three times the third angle.

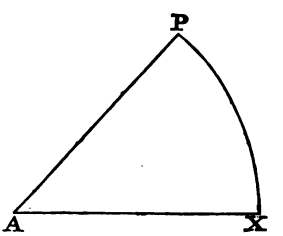
6. The angle contained by the compass-needle with the North, when pointing;—N.E., N.N.W., E.N.E., S.W., S. by E., S.W. by W., E.$\frac{1}{4}$S.

CHAPTER II.

ANGLES AND ARCS.

5 AN angle being of unlimited size, to avoid a difficulty in expressing it by a figure, it is usual to consider an angle as the inclination of a line to one which is in a fixed position, so that **the angle has** always **one fixed side.**

The line AP moves about a point A just as a clock-hand turns round a point, but in the opposite direction. It is always supposed to start from a fixed line AX.



Then the increase and decrease of the angle PAX or XAP (called the angle A) depends wholly on the motion of AP.

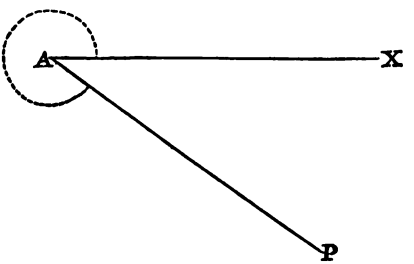
When the angle A is increased, it is because AP has moved away from AX; when it is decreased, it is because AP has moved towards AX.

AP may move on till PAX becomes a right angle, and further, till it becomes an obtuse angle; and since the line AP has a motion of revolution about A which is not limited, it may form two right angles, or three and four, and any number of right angles with AX.

It is to meet this alteration in the idea of an angle that the fixed line AX is always used as one side of the angle.

Euclid does not allow the existence of angles greater than two right angles.

According to his usage, the angle PAX means the small angle, not dotted. But here the angle PAX is



the angle described by the revolution of AP, which has started from the fixed line AX, in the direction opposite to that in which the hands of a clock revolve. So that in this figure PAX is between three and four right angles.

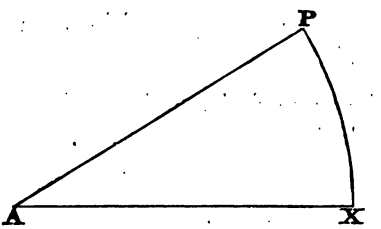
The angle A is made by AP revolving round A in the direction opposite to the hands of a clock, and starting from a fixed line AX.

There is no difficulty or possibility of mistake in speaking of an angle of any size, six or seven right

angles for instance, for the angle A continually increases so long as AP moves thus.

When AP returns to coincidence with AX, four right angles have been traversed, so *to the eye* the angle begins again after four right angles.

6 But as AP revolves, the point P moves along an arc of a circle, and the magnitude of this arc PX is continually increasing as AP moves in the same direction.



It is the simplest way of marking the size and increase of an angle A, to notice the size and increase of the arc PX along which P has moved.

We see that if this arc be doubled, the angle A is doubled; if the arc be bisected, the angle is halved; or whatever multiple or fraction be taken of the arc, the same will be taken of the angle.

So, if we measure the arc, we measure the angle, if the size of the circle be the same. And if all would agree to use a circle of certain constant radius, it would be possible to measure an angle of 1 in. or 1 ft. arc, and to distinguish an angle by that measure. An arc may be taken of any length, which well suggests the descrip-

tion of angles by revolution and the unlimited size of angles.

7 We must then take a **standard arc** as a measure of the arc subtending any angle, and this arc must increase with the circle, for the arcs subtended by any angle increase with that circle.

An arc is taken whose length is equal to the length of the radius, and this is used as the standard arc, with which the arc subtending any angle is compared.

Now there are two propositions on which this method depends.

8 I. In the same circle, angles at the centre vary as the arcs on which they stand (Euc. VI. 33).

II. In different circles, the circumferences are proportional to the diameters.

The latter of these may be proved, by considering the circumference of a circle as the limit of a polygon, whose perimeter approaches the circumference, as the number of the sides is indefinitely increased.

The perimeter of the polygon and therefore of the circle is proportional to its radius.

The arc varies with the angle directly.

So its arc is a good measure of an angle, and the radius is a measure that can well be used to measure the arc, since for the same angle at the centre of different circles, the arc varies as the diameter or radius.

9 **Circular measure is the use of the radius to measure the arc on which an angle stands.**

It measures the angle by the number of radii that are contained in its arc.

If the arc is *twice* the radius, the angle is called 2;

" " *half* " " " " " $\frac{1}{2}$.

If the arc contains π radii, the angle is called π;

" " " θ " " " " θ.

π and θ are **numbers**; θ is an unknown and variable number like x or y; π is the **number of diameters** which are contained in the **circumference** of a circle.

10 The circumference of a circle varies as its diameter, therefore the circumference is a constant multiple of the diameter.

This constant is no exact fraction or integer, it is usually denoted by π, where

$$\pi = 3{\cdot}141592653589793\ldots$$

this is nearly 3·1416, $\frac{22}{7}$ **or** $\frac{355}{113}$.

The latter fraction is easily remembered as the odd numbers 113 355.

11 So there are π diameters, or 2π radii in the circumference of a circle; hence the angle which the whole circumference subtends is called 2π.

The whole circumference subtends four right angles. So the circular measure of four right angles is 2π; and

the circular measure of one right angle is $\frac{\pi}{2}$.

12 The circular measure of an angle is a **number** integral or fractional, namely, the number of radii contained in the arc which subtends an angle, that

is, circular measure is the fraction $\frac{\text{arc}}{\text{radius}}$. **If there are θ radii in the arc, the angle is called θ.**

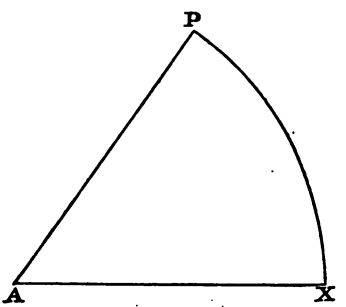
We seem here to be measuring angles without having chosen a standard angle to compare them with. **But the angle subtended by θ radii is θ times the angle subtended by one radius.** So this angle, whose arc is equal to its radius, is really the standard by which the angle θ is measured.

The angle subtended by one radius is called the **Unit of Circular Measure.**

18 **To ascertain the number of degrees or grades in the angle whose circular measure is 1.**

PAX is such an angle, for the arc PX is equal to the radius AP,

$$\therefore \frac{\text{arc}}{\text{radius}} = 1.$$



I. Circumferences of circles vary as their radii; that is,

The circumference contains 2π radii.

The radius is $\frac{1}{2\pi}$ of the circumference.

II. Angles at the centre vary as the arcs on which they stand.

The angle subtended by one radius-length of arc is $\frac{1}{2\pi}$ of the angle subtended by the whole circumference.

But the angle subtended by the whole circumference is four right angles.

The angle subtended by an arc of one radius-length is $\frac{1}{2\pi}$ of four right angles $= \frac{360^\circ}{2\pi} = \frac{180^\circ}{\pi}$.

The **Unit of Circular Measure** (arc = radius)

$$= \frac{180^\circ}{\pi} \text{ or } \frac{200^g}{\pi}.$$

This is about $57^\circ \cdot 29577\ldots$

14 This proof may be shortly written;—

$$\frac{\text{angle PAX}}{\text{four right angles}} = \frac{\text{arc PX}}{\text{circumference}} = \frac{\text{radius}}{2\pi \text{ radii}} = \frac{1}{2\pi},$$

$$\therefore \text{angle PAX} = \frac{180^\circ}{\pi} = 57^\circ \cdot 29577\ldots$$

15 The angle whose circular measure is θ is θ times this angle $= \frac{180\theta^\circ}{\pi}$ or $\frac{200\theta^g}{\pi}$.

Therefore, where D is the number of degrees in the angle whose circular measure is θ, $D = \frac{180\theta}{\pi}$.

And where G is the number of grades in the same angle, $G = \frac{200\theta}{\pi}$.

Both these facts are included in the formulæ;

$$\therefore \frac{D}{9} = \frac{G}{10} = \frac{20\theta}{\pi},$$

where D is the number of degrees,

G is the number of grades,

θ is the number of radii in the arc,

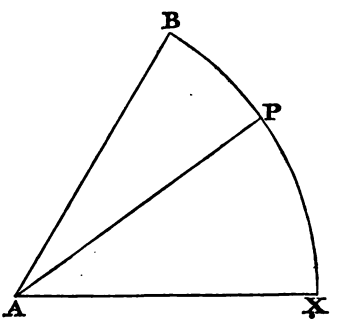
of the same angle.

16 Before proceeding, it will be well to give a **recapitulation** of some important points.

1. An angle is always supposed to be formed by a fixed line, and a line which has revolved about a point in it, in the direction opposite to the hands of a clock.

2. An angle may be measured by the number of right angles or parts of a right angle which it contains.

3. The fraction $\frac{\text{arc}}{\text{radius}}$ is a good measure of an angle; for (Euc. VI. 33) (if BAX is an angle whose arc is equal to the radius)



$$\frac{\text{angle PAX}}{\text{angle BAX}} = \frac{\text{arc PX}}{\text{arc BX}} = \frac{\text{arc PX}}{\text{radius AP}},$$

$$\therefore \text{angle PAX} = \frac{\text{arc}}{\text{radius}} \text{ angle BAX},$$

which shews that the size of the angle depends on $\frac{\text{arc}}{\text{radius}}$, since BAX is a fixed angle.

4. There are π diameters in the circumference of a circle $= 2\pi$ radii or $2\pi r$. $\pi = 3{\cdot}1416$, or $\frac{22}{7}$, or $\frac{355}{113}$ nearly.

5. The complement of an angle is its defect from one right angle. Subtract the angle from 90°, 100^g or $\frac{\pi}{2}$, to find its complement.

6. The supplement of an angle is its defect from two right angles. Subtract the angle from 180°, 200^g or π, to find its supplement.

7. Rule for reduction of the measures of angles,

$$\frac{D}{9} = \frac{G}{10} = \frac{20\theta}{\pi},$$

which gives

$$\left.\begin{aligned} D &= \frac{9}{10} G \text{ or } \frac{180\theta}{\pi} \\ G &= \frac{10}{9} D \text{ or } \frac{200\theta}{\pi} \\ \theta &= \frac{D}{180}\pi \text{ or } \frac{G}{200}\pi \end{aligned}\right\},$$

the operations which find the other two measures when one is given.

The following general examples in angles will suffice to give confidence in the use of this rule.

Obs. As the circular measure represents a pure number, a ratio, it is always placed without any mark of denomination after it, as an angle $\frac{3}{4}$ or $2\frac{1}{2}$; where these numbers are the circular measure.

This is a pure convention, somewhat analogous to our expression "That child is just eight," or "That man is worth a million," meaning, of course, years and pounds sterling.

The other measures must always have their marks added.

PAPERS ON ANGLES.

I. 1. What is the Circular Measure of an Angle?

Give the circular measure of the angle subtended by a circular watch-spring 3 in. long and radius $1\frac{1}{2}$ in.

2. Find the number of degrees in the angles 10^g, 30^g, 25^g, and the number of grades in 45°, 60°, 150°.

3. Find the circular measure of $\frac{1}{8}$ of a right angle; $\frac{2}{3}$ of 4 right angles; one grade; one degree.

4. Add together $33^\circ.45'.21''$; $2^g.22''$; $3^g.21'$;

$$105^g.14'.25''; \quad \cdot 35\pi; \quad \frac{1}{8}; \quad 1\cdot 02\pi.$$

5. What angle in degrees, minutes and seconds is meant by $\theta = 2$?

II. 1. Find the angle whose circular measure is 1, and deduce the number of degrees in 3·05.

2. What number of grades, &c. is made by an arc of 5 chains at the centre of a railway curve, radius 80 chains?

3. Change to circular measure;

$3^{\circ}.14'.25''$; $185^{g}.21^{c}.5^{cc}$; $105^{\circ}.5''$.

4. Through what angle does a steersman move his wheel when he passes n out of the m spokes?

5. What convenience is there in the French system of angle measurement?

III. 1. Prove that an angle may be measured by the fraction $\frac{\text{arc}}{\text{radius}}$, and find the arc which will subtend one degree at the centre of a circle, radius 1 foot.

2. Change to grades the angles; $13^{\circ}.12'$; $406^{\circ}.3''$; and $\frac{\pi}{18}$; $\frac{2\pi}{3}$; 21·02; 2·01.

3. Find the angle in circular measure made by the hands of a clock at $5^{h}.15'$, a quarter to 8, 3.30, 6.5.

4. Prove the formula $\frac{D}{9}=\frac{G}{10}=\frac{20\theta}{\pi}$, giving the meaning of the terms used.

5. Find the circumference of a circle where an angle of 33° is subtended by an arc of 4 inches.

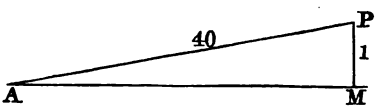
CHAPTER III.

THE TRIGONOMETRICAL RATIOS.

17 THERE remains to be considered, the Trigonometrical measurement of angles, that is, the measurement of angles by triangles.

The inclination of a slope on a railway to the horizon is measured in this way. A railway is said to rise 1 in 40 when for every 40 yards of the railway it rises 1 yard.

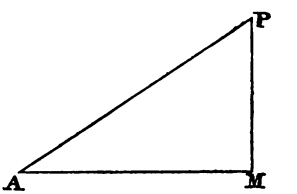
This is an indication of the angle made by the railway and the horizon, and is denoted by saying that the **sine of the angle** of inclination is $\frac{1}{40}$.



The sine of an angle shews its size. If an angle has a sine $\frac{1}{2}$, it represents a slope of 1 in 2, a known slope and angle. If the sine be $\frac{1}{4}$ it represents a slope of 1 in 4. A mistake must here be guarded against. The angle whose sine is $\frac{1}{2}$ is not double the angle

whose sine is $\frac{1}{4}$, as may easily be seen by drawing a figure.

18 AMP is a right-angled triangle, and so APM is the complement of PAM. And it will be seen that $\frac{PM}{AP}$ is the sine of PAM; while $\frac{AM}{AP}$ is the **sine of its complement** APM. This is called the cosine of A, and represents the rate of progress along the horizontal line compared with the progress up the hill. If for every 40 up the hill AP, 13 is gone along the horizon AM, the cosine of PAM or cos A is said to be $\frac{13}{40}$.



19 The values of sine A and cosine A may be made familiar thus:

I go 50 yds. straight up a hill-side AP.

I **rise** PM or 30 yds. This shews that the slope is 3 in 5 and the **sine** of its inclination is $\frac{3}{5}$.

Also I go 40 yds. along a horizontal line **on the map** AM.

This shews me that the **cosine** of the inclination is $\frac{4}{5}$.

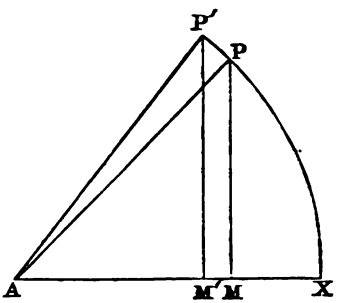
That is, $\sin A = \frac{PM}{AP}$; $\cos A = \frac{AM}{AP}$.

20 We will proceed to discuss these more fully.

An angle is formed by a line starting from a fixed position and revolving in the direction opposite to the hands of a watch.

The point P moves in a circle and PAX increases to P'AX. From P a perpendicular PM is let fall on the fixed line.

As the angle increases, **PM increases**, and **AM diminishes**; while P moves in a circle, that is, AP remains constant.



The ratio of PM or AM to AP will determine the size of the angle. So that if we know the value of $\frac{PM}{AP}$, this will fix the size of the angle. So too if we know the ratio of AM to AP; for instance, if AP = 2AM the size of the angle PAM is fixed.

21 Just as in circular measure we examine the fraction which the arc is of the radius, how many radii or parts of radii there are in the arc, so here we enquire what fractions the perpendicular PM and the base AM are of the radius.

The first of these is called the sine of A.

$\sin A = \frac{PM}{AP}$ and **increases** as the angle increases.

The other is called the cosine of A.

$\cos A = \frac{AM}{AP}$ and **decreases** when the angle increases.

22 There are six other ratios of the angle PAM, which are derived from these, and they are of two kinds; those which **increase** while the angle increases and decrease while it decreases, and those which vary in the opposite way, *viz.* **decrease** while the angle increases.

The first of these are the **sine,** the **tangent,** the **secant, and** the **versed sine of A.**

The others are of the opposite kind, they **decrease** when the angle increases. These are distinguished by the prefix **co,** since they are the corresponding **ratios of the complement** of A; they are the **cosine,** the **cotangent,** the **cosecant,** and the **coversed sine** of A.

23 Now these are all connected with sine and cosine of A in a way that should thus be remembered:

$\sin A = \frac{PM}{AP}$ **increases**
$\cos A = \frac{AM}{AP}$ **decreases**
} tan A and cot A are **ratios,** sec A and cosec A are **reciprocals** of these,

$\tan A = \frac{\sin A}{\cos A} = \frac{PM}{AM}$; $\sec A = \frac{1}{\cos A} = \frac{AP}{AM}$ **increase,**

$\cotan A = \frac{\cos A}{\sin A} = \frac{AM}{PM}$; $\operatorname{cosec} A = \frac{1}{\sin A} = \frac{AP}{PM}$ **decrease,**

$\operatorname{versin} A = 1 - \cos A$; $\operatorname{covers} A = 1 - \sin A$.

It is best to remember the principle of increase and decrease, and so to have the ratios sorted into the

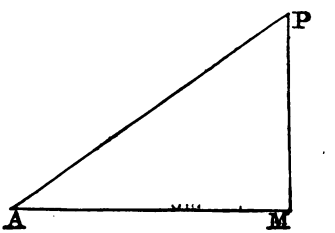
two corresponding sets; the one **direct** and the other **inverse.**

24 The tangent and cotangent of A are the ratios of the sides which contain the right angle. So the numerator must be in one case the increasing and in the other the decreasing side. If a line AM of four parts have a perpendicular PM erected on it of three such parts, an angle will be formed by AP with AM, whose tangent is $\frac{3}{4}$ and cotangent $\frac{4}{3}$.

25 To draw an angle whose sine is $\frac{3}{5}$.

Take a line AP divided into five equal parts, and take a line PM equal to three of these parts.

A circle described on diameter AP in which the line PM is placed will form an angle PAM whose sine is $\frac{3}{5}$.



26 The ratios are all connected with one another by formulæ which must be remembered.

We know that $PM^2 + AM^2 = AP^2$ (Euc. I. 47),

$$\therefore \frac{PM^2}{AP^2} + \frac{AM^2}{AP^2} = 1, \text{ dividing by } AP^2.$$

1. $\therefore \sin^2 A + \cos^2 A = 1,$

and $\frac{PM^2}{AM^2}+1=\frac{AP^2}{AM^2}$, dividing by AM^2.

2. $\therefore \tan^2 A+1=\sec^2 A$,

and $\frac{AM^2}{PM^2}+1=\frac{AP^2}{PM^2}$, dividing by PM^2.

3. $\therefore \cot^2 A+1=\operatorname{cosec}^2 A$.

On the use of these three formulæ and the original connections of tan A and cot A with sin A and cos A, and of sec A and cosec A with sin A and cos A, some exercises will be given.

In proving these identities, one side should be taken, and reduced to the form of the other by the aid of these three formulæ and the original connections.

(1) Example. Prove $\tan^2 A=\sec^2 A-1$.

$$\tan^2 A=\frac{\sin^2 A}{\cos^2 A}$$

$$=\frac{1-\cos^2 A}{\cos^2 A}=\frac{1}{\cos^2 A}-1$$

$$=\sec^2 A-1;$$

$\therefore \tan^2 A=\sec^2 A-1$. Q. E. D.

EXERCISES ON THE RATIOS.

Prove the following identities:

I. 1. $\sin^2 A=1-\cos^2 A$.

2. $\sec^2\theta=\frac{1}{1-\sin^2\theta}$.

3. $\tan^2 A=\frac{1-\cos^2 A}{\cos^2 A}$.

4. $\cot^2\theta = \operatorname{cosec}^2\theta - 1$.

5. $\cos A = \cot A \sin A$.

6. $\cot^2\theta = \dfrac{\cos^2\theta}{1-\cos^2\theta}$.

7. $\cos^2\theta = \dfrac{1}{1+\tan^2\theta}$.

8. $\sin^2\theta = \dfrac{\tan^2\theta}{1+\tan^2\theta}$.

9. $\sec A = \tan A \operatorname{cosec} A$.

10. $\sec^2 A = \dfrac{\operatorname{cosec}^2 A}{\operatorname{cosec}^2 A - 1}$.

II. 1. $\sec A + 1 = \dfrac{1+\cos A}{\cos A}$.

2. $\tan A = \sin A \,.\, \sec A$.

3. $1 + \cos A = \dfrac{\sin^2 A}{1-\cos A}$.

4. $\sec^2 A = 1 + \dfrac{\sin^2 A}{\cos^2 A}$.

5. $\sec^2\theta + \operatorname{cosec}^2\theta = \sec^2\theta \,.\, \operatorname{cosec}^2\theta$.

6. $\cot^2\phi + \tan^2\phi = \sec^2\phi \operatorname{cosec}^2\phi - 2$.

7. $1 + \tan A = \sqrt{\sec^2 A + 2\tan A}$.

8. $\cos^2 A = \dfrac{\cot^2 A}{1+\cot^2 A}$.

9. $(1-\sin A)\sec A = \dfrac{\cos A}{1+\sin A}$.

10. $\sin^4\theta - \cos^4\theta = \sin^2\theta - \cos^2\theta$.

III. (1) Find all the ratios of the angles

1. whose sine $= \frac{1}{2}; \frac{1}{\sqrt{2}}; \frac{3}{2}; \frac{3}{5}; a; \frac{a}{b}$.

2. whose cosine $= 1; \frac{1}{3}; \frac{\sqrt{3}}{2}; \sqrt{\frac{1}{3}}; b^2; a^2 - b^2$.

(2) Find $\sin A$ and $\cos A$, and thence find all the ratios of the different angles A, where

1. $\sec A = 2$; $\cot A = \frac{1}{3}$; $\operatorname{cosec} A = \sqrt{2}$.

2. $\tan A = 0$; $\sec A = 200$; $\tan A = \frac{4}{3}$.

(3) Find $\sec\theta$ and $\operatorname{cosec}\theta$, and thence deduce all the ratios of θ, where

1. $\tan\theta = \cdot 02$; $\cot\theta = 20$; $\cos\theta = \frac{1}{2}$.

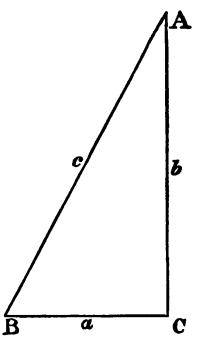
2. $\cot\theta = \cdot 125$; $\tan\theta = 4 \cdot 3$; $\sin\theta = \frac{3}{4}$.

CHAPTER IV.

TRIANGLES.

27 To shew the immediate use of these ratios it will be seen that it is with these measures of angles and not with the arithmetical measures that the sides of triangles are connected.

Let ABC be a right-angled triangle. We may denote the angles by A, B, C, and the sides opposite to them by a, b, c.



Here A, B, C represent the number of degrees, grades, or the circular measure of the angles, and a, b, c denote the number of feet, inches or any measure of the length of the sides. So that these letters, as is usual in Algebra, represent **numbers.**

AC is called the side b, being opposite to the angle B.

Then $$\frac{b}{c} = \frac{AC}{AB} = \sin B; \quad \therefore\ b = c \sin B.$$

and $$\frac{a}{c} = \frac{BC}{AB} = \cos B; \quad \therefore\ a = c \cos B,$$

or $$= \sin A, \quad \text{or} \quad = c \sin A.$$

In the same way we obtain all the values;

$a = c \sin A$, or $c \cos B$, and also $= b \tan A$, or $b \cot B$.

$b = c \cos A$, or $c \sin B$, and also $= a \cot A$, or $a \tan B$.

$c = a \operatorname{cosec} A$, or $a \sec B$, and also $= b \sec A$, or $b \operatorname{cosec} B$.

These are true for the case of a right-angled triangle.

These values of the sides should be written out from the figure alone, over and over again, so as to make the results and the method of obtaining them familiar.

28 *Also in any triangle*

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

This may be easily proved for an acute-angled triangle.

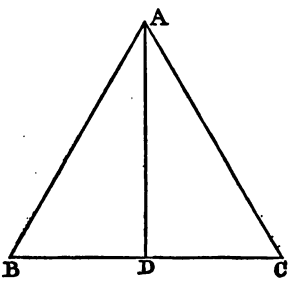
Draw a perpendicular AD from A on BC,

$$AD = b \sin C$$

and also $= c \sin B$;

$$\therefore b \sin C = c \sin B$$

and $$\frac{\sin B}{b} = \frac{\sin C}{c}.$$



And by drawing a perpendicular from C we may prove

$$b \sin A = a \sin B; \quad \therefore \frac{\sin A}{a} = \frac{\sin B}{b};$$

$$\therefore \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

29 *Also in any triangle*

$$a = b\cos C + c\cos B,$$

for $a = DC + BD,$

while $DC = b\cos C$ and $BD = c\cos B;$

$$\therefore\ a = b\cos C + c\cos B.$$

Similarly $b = c\cos A + a\cos C,$

$$c = a\cos B + b\cos A.$$

30 **These formulæ may be used to find values of parts of a triangle when others are given.**

A figure should always be drawn and the values which are given should be written by the corresponding sides or angles of the triangle.

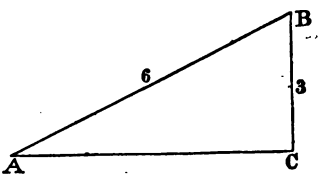
Examples :—

I. In a right-angled triangle,

Given $a = 3$, $\operatorname{cosec} A = 2$ } Find c.

$\therefore \sin A = \frac{1}{2}$, $\cos B = \frac{1}{2}$,

and $c = a \operatorname{cosec} A = 6$.



II. In any triangle,

given $\sin A = \cdot 3$, $\sin B = \cdot 12$, $b = 3$,

we know that

$$\frac{\sin A}{a} = \frac{\sin B}{b},\ \therefore \frac{\cdot 3}{a} = \frac{\cdot 12}{3},\ \therefore a = \frac{15}{2}.$$

When one ratio of an angle is given, we have seen that any of the others can be found from it. But this does not imply that only one sort of ratio is necessary and the rest are superfluous, but the richness of the forms found for the sides of triangles shews that all are necessary because of the large choice of forms which they give for practical use.

EXAMPLES ON TRIANGLES.

I. In a right-angled triangle where C is the right angle—

1. $\sin A = \cdot 4$; $c = 60$ ft. Find a and b.
2. $\sec A = 2$; $c = 20$ in. Find a, b and $\sin B$.
3. $\cos B = \frac{1}{2}$; $c = 45$. Find a, b and $\tan A$.
4. $\operatorname{cosec} B = \sqrt{3}$; $c = 23$. Find a, b and $\sec A$.
5. $\tan B = \sqrt{3}$; $c = 1$. Find a, b and $\cos A$.

II. In any triangle—

1. $\sin A = \frac{\sqrt{3}}{2}$; $\cos B = \frac{\sqrt{3}}{2}$; $b = 40$. Find a.
2. $\sin A = \cdot 5$; $\sin B = \cdot 2$; $a = 25$. Find b.
3. $\sin C = \cdot 2$; $\sin A = \cdot 3$; $a = 18$. Find c.
4. $\sin B = \frac{1}{2}$; $\cos C = \frac{1}{3}$; $b = 30$. Find c.
5. $\cos A = \frac{3}{5}$; $\cos B = \frac{3}{5}$; $b = 10$. Find a.

III. Where C is a right angle—

1. $a = 30$; $b = 20$. Find sin A and tan B.
2. $a = 25$; $b = 2$. Find sin B and sin A.
3. $c = 13$; $b = 12$. Find sin B and cot A.
4. $a = 5$; $b = 12$. Find cosec B and cot A.
5. $b = 12$; $c = 50$. Find cot B and cot A.

These may serve as an exercise here or afterwards in the measurement of triangles. We have treated at present of angles less than a right angle alone, so acute-angled and right-angled triangles alone are dealt with. But as the formulæ are the same for all triangles, what has been said here of the formula given need not be repeated for obtuse-angled triangles.

CHAPTER V.

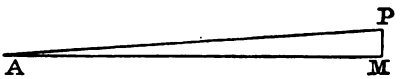
THE RATIOS OF CERTAIN ANGLES.

31 The treatment of triangles with which the last chapter ended shewed that if the trigonometrical ratios of the angles of a triangle be known, the sides of the triangle may be found by the use of them, if one side is known.

Tables are composed, which give the trigonometrical ratios of all angles, but there are some angles of which it is easy to find and remember the ratios.

32 *The angle* 0.

The sine of the small angle A is $\frac{PM}{AP}$.



When the angle A vanishes PM vanishes, so that we may say PM = 0, and sin 0 = 0.

And when A vanishes, AP coincides with AM, and is equal to it, so that $\cos A = \frac{AM}{AP} = 1$, giving $\cos 0 = 1$.

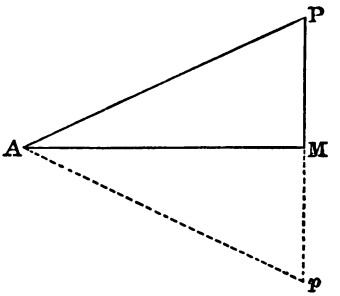
33 *An angle of* 90°, *or* $\frac{\pi}{2}$.

These are the measures of a right angle.

When PAM becomes a right angle, PM, the perpendicular, coincides with AP and is equal to it; while AM vanishes; giving $\sin \frac{\pi}{2} = 1$, $\cos \frac{\pi}{2} = 0$.

34 *An angle of* 30°, or $\frac{\pi}{6}$.

Let the line AP revolve from AM till PAM is $\frac{1}{3}$ of a right angle.



Draw a perpendicular PM from P. Then the triangle PAM is half an equilateral triangle, of which AP is one side, and AM is the perpendicular on the base.

Then $MP = \frac{1}{2}AP$, $\quad \sin 30^\circ = \frac{PM}{AP} = \frac{1}{2}$.

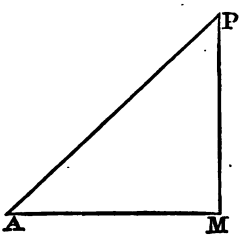
Also $\quad \sin^2 30^\circ + \cos^2 30^\circ = 1$;

$$\therefore \cos^2 30^\circ = \frac{3}{4}, \quad \cos 30^\circ = \frac{\sqrt{3}}{2};$$

$$\therefore \sin 30^\circ = \frac{1}{2}, \quad \cos 30^\circ = \frac{\sqrt{3}}{2}.$$

35 *An angle of* 45°, or $\frac{\pi}{4}$.

The angle PAM is formed by AP with the fixed side AM from which it has revolved through half a right angle. PAM being half a right angle, APM must also be half a right angle,



$$\therefore \text{AM} = \text{PM},$$

and $$\text{PM}^2 + \text{AM}^2 = \text{AP}^2 \text{ (Euc. I. 47)};$$

$$\therefore 2\text{PM}^2 = \text{AP}^2;$$

$$\sin^2 45^\circ = \left(\frac{\text{PM}}{\text{AP}}\right)^2 = \frac{1}{2};$$

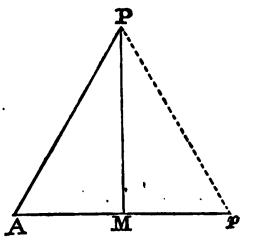
$$\therefore \sin 45^\circ = \frac{1}{\sqrt{2}},$$

and for the same reasons

$$\cos 45^\circ = \frac{1}{\sqrt{2}},$$

since PM = AM.

36 *An angle of* 60°, or $\frac{\pi}{3}$.



If the line AP revolve from AM till PAM is two-thirds of a right angle.

Draw a perpendicular PM from P on AM. Then the angle PAM is the angle of an equilateral triangle, which may be completed, having AP as one of its sides, and PM as the perpendicular on the base;

$$\therefore \text{AM} = \frac{1}{2}\text{AP} \text{ and } \cos 60^\circ = \frac{\text{AM}}{\text{AP}} = \frac{1}{2}.$$

Also $$\sin^2 60^\circ + \cos^2 60^\circ = 1;$$

$$\therefore \sin^2 60^\circ = \frac{3}{4}, \quad \sin 60^\circ = \frac{\sqrt{3}}{2};$$

and $$\sin 60^\circ = \frac{\sqrt{3}}{2}, \quad \cos 60^\circ = \frac{1}{2}.$$

37 These results may be gathered together,

$$\sin 0=0;\ \sin 30^\circ=\frac{1}{2};\ \sin 45^\circ=\frac{1}{\sqrt{2}};\ \sin 60^\circ=\frac{\sqrt{3}}{2};\ \sin 90^\circ=1.$$

$$\cos 0=1;\ \cos 30^\circ=\frac{\sqrt{3}}{2};\ \cos 45^\circ=\frac{1}{\sqrt{2}};\ \cos 60^\circ=\frac{1}{2};\ \cos 90^\circ=0.$$

These must all be remembered, and to help the memory it is usual to place them in a more convenient form,

	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin$	$\sqrt{0},$	$\sqrt{\frac{1}{4}},$	$\sqrt{\frac{1}{2}},$	$\sqrt{\frac{3}{4}},$	$\sqrt{1}.$
$\cos$	$\sqrt{1},$	$\sqrt{\frac{3}{4}},$	$\sqrt{\frac{1}{2}},$	$\sqrt{\frac{1}{4}},$	$\sqrt{0}.$

All the numbers are under a square root, and by arranging them thus, it is easier to remember the results as "*The square root of the fractions* $0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$," than to remember the series $0, \frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{2}, 1$, which does not so easily strike the eye as being in order of magnitude.

38 Having these values of $\sin A$ and $\cos A$, the other ratios must be deduced from them, not remembered, viz.

$$\tan 60^\circ=\frac{\sin 60^\circ}{\cos 60^\circ}=\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}=\sqrt{3}.$$

$$\sec\frac{\pi}{4}=\frac{1}{\cos\frac{\pi}{4}}=\frac{1}{\frac{1}{\sqrt{2}}}=\sqrt{2}.$$

This recalls to mind the fact that if one of the ratios of an angle be known, all the other ratios of that angle may be deduced from it. Yet of these particular angles, it is usual to remember both the sine and the cosine, since they may be represented in so simple a form; and since the other ratios follow at once, as ratios and reciprocals of these.

39. Three sets of formulæ have been found, for deducing sides of triangles from certain known parts which are given; viz.—

I. In a right-angled triangle

$$b = c \cos A \text{ and the cognate formulæ.}$$

II. In any triangle

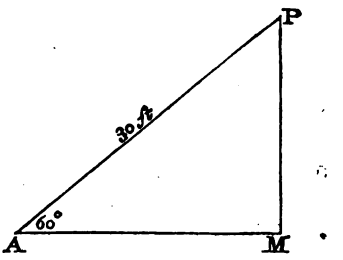
$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

III. Also $a = b \cos C + c \cos B$,
with two similar formulæ.

These formulæ connect the sides of a triangle with the ratios of the angles.

If the angles given are those of which we have just treated, viz. 0, 30°, 45°, 60°, 90°, we know their ratios, and we can use them when required. If other angles are used the ratios must be discovered elsewhere, but if one of the ratios be given, any of the others may be found. It is well always to draw a figure carefully, and to write by the sides their lengths and in the angles their magnitude.

Ex. I. A ladder 30 feet long reaches to the top of a wall, and is inclined at an angle 60° to the street. How high is the wall?



PM is the wall.

$$PM = AP \sin 60^\circ$$
$$= 30 \times \frac{\sqrt{3}}{2} \text{ ft.}$$
$$= 15\sqrt{3} \text{ ft.}$$

Now $\sqrt{3}$ is nearly 1·7320, so this may be reduced to feet and inches = 25 ft. 11·76 in.

Ex. II. If sin A = ·02. Find cot A.

$$\text{cosec } A = 50,$$
$$\cot^2 A = 1 + \text{cosec}^2 A = 2501;$$
$$\therefore \cot A = \sqrt{2501}.$$

EXAMPLES ON THE USE OF THE TRIGONOMETRICAL RATIOS.

I. 1. What is the sine of the angle made with the ground by a ladder 60 ft. long when it rests on the top of a wall 30 ft. high?

2. Find $\sec 60^\circ$; $\tan \frac{\pi}{6}$; $\cot \frac{\pi}{3}$; $\sin 50^\circ$.

3. Find the distance between the top of a mountain (1,000 yds. high) and a village from which the top appears at an angle of elevation whose sine is ·02.

4. Find $\tan \frac{\pi}{4}$ and $\sec \frac{\pi}{3}$, cosec 50° and cot 60°.

5. Find the angle between a stick 20 in. long, and the ground on which its shadow from a vertical sun is 10 in.

6. Prove that $\sin \frac{\pi}{6} + \cos \frac{\pi}{3} = \sin^2 \frac{\pi}{6} + \cos^2 \frac{\pi}{6}$.

7. Find the depth of a pit which is reached by a tunnel 500 yards long, inclined to the horizon at an angle 30°.

8. If $\sin A = \cdot 3$, find sec A, cosec A, and cot A.

II. 1. Two straight roads separate at an acute angle. When I have gone 2 miles along one, I am 1 mile from the other. Find the angle between them.

2. Write out all the trigonometrical ratios of the angle 30°, and of the angle whose cosine is $\frac{1}{3}$.

3. Find the side b of a triangle where $a = 3$, $\cos A = \cdot 2$ and $\cos B = \cdot 3$.

4. What is the height of a spire from which a rope 400 feet long to the ground makes an angle of 45° with it?

5. In a triangle, $\cos A = \frac{1}{2}$ and $c = 90^\circ$. Find a when $c = 30$ ft., or when $b = 20$ ft.

6. Find the value of $\sin 30^\circ + \cos 45^\circ + \tan 45^\circ$, and take its square root.

7. How far off is a church tower (200 feet high) which subtends at my eye an angle whose tangent is $\cdot 01$?

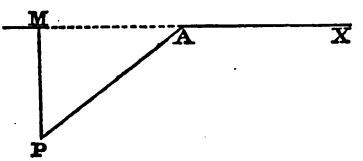
CHAPTER VI.

RATIOS OF LARGE ANGLES.

40 WE have examined the nature of the Trigonometrical ratios, sine, cosine &c., and their uses. We have seen that for certain angles the ratios were easily found.

The angles there discussed were all less than a right angle; it remains to extend the consideration to angles greater than a right angle.

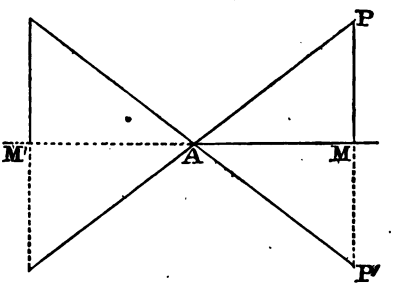
An angle is formed by a straight line starting from a fixed position and revolving in the direction opposite to the hands of a watch.



AP starts from AX and revolves.

Whatever angle AP makes with AX, **a perpendicular PM is drawn from P upon the fixed side AX;** $\frac{PM}{AP}$ and $\frac{AM}{AP}$ are called the **sine** and **cosine** of PAX, and the other ratios are connected as before with them.

41 During the revolution of AP through four right angles there are **four positions,** where the ratios have the **same numerical values** as for any acute angle PAM, namely, the four positions shewn in the figure.



With the increase of PAM, AM diminishes, and on the left of A it is less than 0, which value it had at A; therefore all lines along AM which lie to the left of A are considered as less than zero, that is as negative.

In the same way PM, when PAM diminishes, becomes less, and when PM is below the line AM, it is less than 0, that is negative.

42 These are the **conventions** adopted with regard to the **signs of lines** that **all parts of the fixed line AX which lie to the left of A are negative; all to the right are positive in sign.**

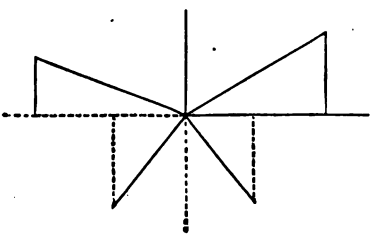
All lines perpendicular to AX which lie below it are negative, and above it are positive.

To these may be added the **convention** adopted with respect to **angles.**

If an angle be formed by the revolution of AP in the direction opposite to the hands of a clock it is positive, but if **in the same direction it is negative.**

43 In accordance with these we may trace the changes in the value of sin A and cos A during the description of four right angles.

In the first quadrant $\left(0 \text{ to } \frac{\pi}{2}\right)$, all acute angles.



All negative lines are dotted.

PM and AM are both positive,

sin A increases from 0 to 1,

cos A decreases from 1 to 0.

In the second quadrant $\left(\frac{\pi}{2} \text{ to } \pi\right)$, one to two right angles.

PM is positive and AM negative,

sin A decreases from 1 to 0,

cos A decreases from 0 to -1.

In the third quadrant $\left(\pi \text{ to } \frac{3\pi}{2}\right)$, two to three right angles.

PM and AM are both negative.

sin A decreases from 0 to -1,

cos A increases from -1 to 0.

In the fourth quadrant $\left(\frac{3\pi}{2} \text{ to } 2\pi\right)$, three to four right angles.

PM is negative and AM positive,

sin A increases from -1 to 0,

cos A increases from 0 to 1.

44 We see then that the ratios are not the same for any angles except those which differ by a multiple of four right angles.

For the **first** quadrant is the only one where **all** the ratios are **positive.**

In the **second,** cos A is negative, so that tan A and cot A are both negative and **sin A** and **cosec A** alone are **positive.**

In the **third,** sin A and cos A are both negative, **tan A** and **cot A** alone are **positive.**

In the **fourth,** sin A is negative, **cos A** and **sec A** alone are **positive,** since tan A and cot A will be negative.

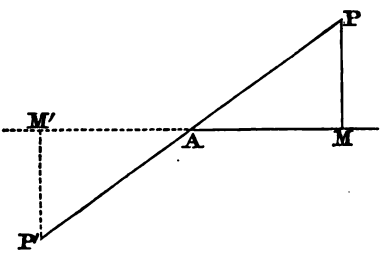
But the ratios of angles greater than a right angle are obviously **connected with those of acute angles,** since they are the ratios of the sides of a right-angled triangle, though they **differ in sign** from them.

The following sections will shew how they are connected.

45 I. **The ratios of an angle are unaltered by the subtraction of any multiple of four right angles from it,** for the revolution of a line through four right angles does not alter its position.

46 II. **An angle which differs from another, by two right angles, has numerically the same sine and cosine, but with opposite signs.**

Let AP revolve till it is in a straight line AP′ with its former position AP, that is, let it revolve through two right angles.



Then the whole angle P′AM is equal to PAM together with two right angles, and if PAM be denoted by α, $P'AM = \pi + \alpha$.

Draw a perpendicular P′M′ from P′ on AM produced.

$$\frac{P'M'}{AP'} = \sin(\pi + \alpha) \text{ and } \frac{PM}{AP} = \sin\alpha.$$

$\frac{PM}{AP}$ is numerically equal to $\frac{P'M'}{AP'}$,

but PM and P′M′ are of opposite sign;

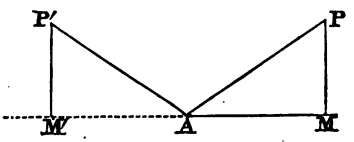
$$\therefore \frac{P'M'}{AP'} = -\frac{PM}{AP}, \text{ that is } \sin(\pi + \alpha) = -\sin\alpha.$$

Similarly $\frac{AM'}{AP'} = -\frac{AM}{AP}$ *i.e.* $\cos(\pi+\alpha) = -\cos\alpha$;

$$\left.\begin{aligned}\therefore \sin(\pi+\alpha) &= -\sin\alpha,\\ \cos(\pi+\alpha) &= -\cos\alpha;\end{aligned}\right\}$$

from which it will be seen that **all the ratios change their sign except tan $(\pi+\alpha)$ and cot $(\pi+\alpha)$.**

47 III. **The supplement of an angle has numerically the same sine and cosine as the angle itself, but the cosine is of the opposite sign.**



If the line AP revolve to AP' so that it makes P'AM the supplement of PAM, *i.e.* so that the angle PAM together with P'AM will make up two right angles, then PAM = P'AM'.

Draw a perpendicular P'M' from P' on AM produced.

$$\frac{P'M'}{AP'} = \sin(\pi-\alpha) \text{ and } \frac{PM}{AP} = \sin\alpha.$$

And $$\frac{P'M'}{AP'} = \frac{PM}{AP},$$

since PM and P'M' are of the same sign,

$$\therefore \sin(\pi-\alpha) = \sin\alpha.$$

Similarly $\frac{AM'}{AP'} = -\frac{AM}{AP}$, $\therefore \cos(\pi - \alpha) = -\cos\alpha$,

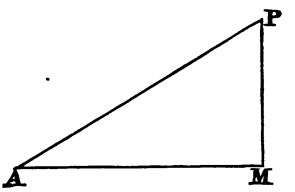
since AM and AM′ are of opposite signs.

$$\left.\begin{aligned} \therefore \sin(\pi - \alpha) &= \sin\alpha, \\ \cos(\pi - \alpha) &= -\cos\alpha, \end{aligned}\right\}$$

from which it will be seen that **all the ratios of the supplement are of opposite sign except sin $(\pi - \alpha)$ and cosec $(\pi - \alpha)$.**

48 IV. **The sine, tangent, and secant of an angle are the cosine, cotangent, and cosecant of its complement.**

This was pointed out in the first mention of the ratios. For the two angles APM and PAM of any right-angled triangle are complementary, and $\frac{PM}{AP}$, which is the sine of the angle at A is the cosine of the angle at P, and all the ratios of A are the complementary ratios of P.



Therefore $\sin 65^\circ = \cos 25^\circ$,

$\operatorname{cosec} 75^\circ = \sec 25^\circ$,

or $\tan\frac{\pi}{6} = \cot\frac{\pi}{3}$.

49 Owing to this connection between the ratios of angles it is unnecessary to give tables of ratios for angles greater than 45°; for ratios of all angles may be **reduced to ratios of angles less than 45°** by these rules:—

I. Subtract any multiple of four right angles, 360°, 400^g, or 2π, without altering the ratios. Thus

$$\tan 668^\circ = \tan 308^\circ.$$

II. Subtract 180°, 200^g, or π *from it*, keeping the same ratio but changing the signs of all but the tangent and cotangent. Thus $\tan 380^\circ = \tan 128^\circ$.

III. Subtract *it from* 180°, 200^g, π; the ratios remain the same, but all change their sign except the sine and cosecant. Thus $\tan 128^\circ = -\tan 52^\circ$.

IV. Subtract it from 90°, 100^g, $\frac{\pi}{2}$; changing all ratios to complementary ratios. Thus

$$-\tan 52^\circ = -\cot 38^\circ;$$

therefore $$\tan 668^\circ = -\cot 38^\circ.$$

50 A few examples are given in the use of these: it is well to **keep the figures** given in cases II. and III. **in the mind** when using those methods, as they give the **true sign by inspection.**

The following examples should be done in this way, by using the four rules one by one: for example,

$$\sin 235^\circ = -\sin 55^\circ = -\cos 35^\circ,$$

$$\operatorname{cosec} 655^g = \operatorname{cosec} 255^g = -\operatorname{cosec} 55^g = -\sec 45^g,$$

$$\tan \frac{15}{3}\pi = \tan \frac{4}{3}\pi = \tan \frac{\pi}{3}.$$

Reduce to ratios of angles less than half a right angle:—

(1) $\sin 832^\circ$. (2) $\sin 553^g$. (3) $\sin 862^g$. (4) $\cos 322^\circ$.

(5) $\operatorname{cosec} \frac{15}{2}\pi$. (6) $\cos 568^\circ$. (7) $\cos 348^\circ$. (8) $\tan 420^\circ$.

(9) $\cos \frac{13}{3}\pi$. (10) $\cot 438^\circ$. (11) $\tan 482^g$. (12) $\sec 222^\circ$.

(13) $\cos 436^\circ$. (14) $\tan 853^\circ$. (15) $\sin 112^g$. (16) $\sin 1082^\circ$.

(17) $\sec 92^\circ$. (18) $\cot 580^\circ$. (19) $\operatorname{cosec} 765^g$. (20) $\sec 810^\circ$.

CHAPTER VII.

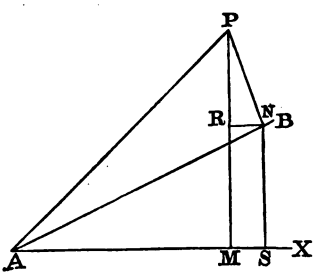
RATIOS OF ADDED ANGLES.

51 *To find the value of* $\sin(A+B)$ *and* $\cos(A+B)$.

Let BAX be represented by A, then if the angle PAB be represented by B, the line PA has revolved from AX through an angle $A+B$.

Take a point P in this **final line AP.**

Draw perpendiculars from P; PN on AB and PM on AX, and perpendiculars NR, NS on PM and AX.



The angle MPN is equal to the angle BAX, for the lines MP, PN which contain it are perpendicular to the lines BA, AX which contain A.

$$\text{Now } \sin(A+B) = \frac{PM}{AP} = \frac{RM+PR}{AP} = \frac{NS+PR}{AP}$$

$$= \frac{NS}{\cdot} \cdot \frac{\cdot}{AP} + \frac{PR}{\cdot} \cdot \frac{\cdot}{AP},$$

where some line **connecting NS and AP** is placed in numerator and denominator, viz. AN; and PN **connecting PR and AP.**

$$\sin(A+B) = \frac{NS}{AN} \cdot \frac{AN}{AP} + \frac{PR}{PN} \cdot \frac{PN}{AP}$$

$$= \sin A \cos B + \cos A \sin B.$$

$$\text{Also } \cos(A+B) = \frac{AM}{AP} = \frac{AS-MS}{AP} = \frac{AS}{AP} - \frac{RN}{AP}$$

$$= \frac{AS}{AN} \cdot \frac{AN}{AP} - \frac{RN}{PN} \cdot \frac{PN}{AP}$$

$$= \cos A \cos B - \sin A \sin B.$$

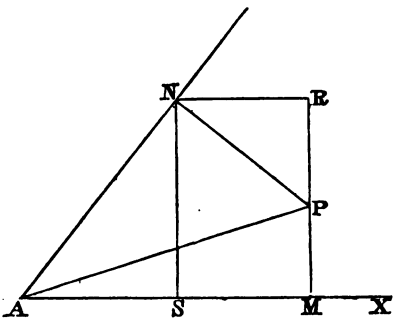
52 *To find* $\sin(A-B)$ *and* $\cos(A-B)$.

Let $NAX = A$ and $PAN = B$.

Let the line OP revolve through an angle A and then in the opposite direction through an angle B.

When it arrives at the position AP it has described an angle $A-B$, or PAX.

Take P in this **final line AP.**



Draw perpendiculars, as before, from P on AN and AX, and from N on PM and AX.

$$\sin(A-B) = \frac{PM}{AP} = \frac{RM-RP}{AP}$$

$$= \frac{NS}{\quad}\cdot\frac{\quad}{AP} - \frac{RP}{\quad}\cdot\frac{\quad}{AP}$$

$$= \frac{NS}{AS}\cdot\frac{AS}{AP} - \frac{RP}{PN}\cdot\frac{PN}{AP},$$

and RPN = NAX = A, since NP, RP are perpendicular to AN and AX.

$$\therefore \sin(A-B) = \sin A\cos B - \cos A\sin B.$$

Also $\cos(A-B) = \frac{AM}{AP} = \frac{AS+SM}{AP}$

$$= \frac{AS}{\quad} \cdot \frac{\quad}{AP} + \frac{NR}{\quad} \cdot \frac{\quad}{AP} = \frac{AS}{AN} \cdot \frac{AN}{AP} + \frac{NR}{PN} \cdot \frac{PN}{AP}$$

$$= \cos A \cos B + \sin A \sin B.$$

53 These four values must be remembered:

$$\sin(A+B) = \sin A \cos B + \cos A \sin B,$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B,$$

$$\sin(A-B) = \sin A \cos B - \cos A \sin B,$$

$$\cos(A-B) = \cos A \cos B + \sin A \sin B.$$

Thus $\cos 75^\circ = \cos(45^\circ + 30^\circ)$

$$= \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}}.$$

Also

$$\sin 15^\circ = \sin(45^\circ - 30^\circ) = \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}},$$

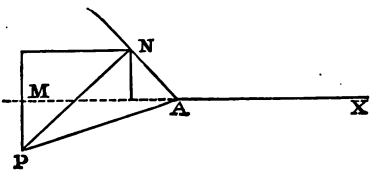
i.e. $\cos 75^\circ = \sin 15^\circ$, which we know, because the one is the complement of the other.

It will be an exercise to find $\sin 75^\circ$ and $\cos 15^\circ$ in the same way.

54 The angles A and B are in the figure given as less than a right angle; the result is in no way altered by drawing any angles.

Thus in the case where A lies between 135° and 180° and $90^\circ > B > 45^\circ$.

$$\sin(A+B) \text{ is represented by } \frac{PM}{AP}.$$



It must be remembered distinctly that the ratios of the **whole angles with their proper signs** must be used according to our conventions.

The figure is formed as before by drawing perpendiculars from P in the **final position of AP**, and completing the figure as before by drawing perpendiculars from N.

Here we see that the value of the conventions of sign is that it makes our formulæ hold generally.

55 *To find values of* $\sin 2A$ *and* $\cos 2A$.

Let $A = B$ in § 51, then $\sin 2A = \sin(A + A)$,

$$\therefore \sin 2A = 2 \sin A \cos A,$$

and $\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A = 2\cos^2 A - 1$.

56 *To find values of* $\sin\frac{A}{2}$ *and* $\cos\frac{A}{2}$ *in terms of* $\sin A$ *and* $\cos A$.

$$\cos A = 1 - 2\sin^2\frac{A}{2} \text{ or } = 2\cos^2\frac{A}{2} - 1,$$

$$\therefore \sin^2\frac{A}{2} = \frac{1-\cos A}{2} \text{ and } \cos^2\frac{A}{2} = \frac{1+\cos A}{2},$$

$$\therefore \sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}} \text{ and } \cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}.$$

These are the values in terms of cos A.

57 Again we have in § 55

$$2\sin\frac{A}{2}\cos\frac{A}{2} = \sin A,$$

and always $\sin^2\frac{A}{2} + \cos^2\frac{A}{2} = 1,$

$$\therefore \sin^2\frac{A}{2} \pm 2\sin\frac{A}{2}\cos\frac{A}{2} + \cos^2\frac{A}{2} = 1 \pm \sin A.$$

The left-hand part is a *square,* so taking the root of both sides,

$$\sin\frac{A}{2} + \cos\frac{A}{2} = \pm\sqrt{1+\sin A},$$

$$\sin\frac{A}{2} - \cos\frac{A}{2} = \pm\sqrt{1-\sin A}.$$

$$\therefore 2\sin\frac{A}{2} = \pm\sqrt{1+\sin A} \pm \sqrt{1-\sin A},$$

$$2\cos\frac{A}{2} = \pm\sqrt{1+\sin A} \mp \sqrt{1-\sin A}.$$

These are the values in terms of sin A.

There are four fundamental formulæ,

58
$$(1)\quad \begin{cases} \sin(A+B) = \sin A\cos B + \cos A\sin B, \\ \sin(A-B) = \sin A\cos B - \cos A\sin B, \\ \cos(A+B) = \cos A\cos B - \sin A\sin B, \\ \sin(A-B) = \cos A\cos B + \sin A\sin B; \end{cases}$$

and from these are derived by addition and subtraction,

59
$$(2)\quad \begin{cases} \sin(A+B) + \sin(A-B) = 2\sin A\cos B, \\ \sin(A+B) - \sin(A-B) = 2\cos A\sin B, \\ \cos(A+B) + \cos(A-B) = 2\cos A\cos B, \\ \cos(A+B) - \cos(A-B) = -2\sin A\sin B; \end{cases}$$

and by writing α for $A+B$ and β for $A-B$,

60
$$(3)\quad \begin{cases} \sin\alpha + \sin\beta = 2\sin\dfrac{\alpha+\beta}{2}\cos\dfrac{\alpha-\beta}{2}, \\ \sin\alpha - \sin\beta = 2\cos\dfrac{\alpha+\beta}{2}\sin\dfrac{\alpha-\beta}{2}, \\ \cos\alpha + \cos\beta = 2\cos\dfrac{\alpha+\beta}{2}\cos\dfrac{\alpha-\beta}{2}, \\ \cos\alpha - \cos\beta = -2\sin\dfrac{\alpha+\beta}{2}\sin\dfrac{\alpha-\beta}{2}. \end{cases}$$

These formulæ should be worked out from the original four, and their derivation and use should be thoroughly known before proceeding.

By the first set of formulæ we can find

61 $\sin 75^\circ$ and $\sin 15^\circ$.

By the second set such results as

$$-\sin A\sin B = \frac{1}{2}\{\cos(A+B) - \cos(A-B)\},$$

By the third set we obtain

$$\sin 3A + \sin 5A = 2 \sin 4A \cos A .$$

~~are obtained.~~

In doing the examples that follow, **one side** of the identical equation **must be taken,** and reduced to the other, not both worked at once;

thus
$$\tan \frac{A}{2} = \frac{\sin \frac{A}{2}}{\cos \frac{A}{2}} = \frac{\sqrt{1 - \cos A}}{\sqrt{1 + \cos A}}$$

$$= \frac{1 - \cos A}{\sqrt{1 - \cos^2 A}} = \frac{1 - \cos A}{\sin A};$$

$$\therefore \tan \frac{A}{2} = \frac{1 - \cos A}{\sin A}.$$

EXAMPLES ON THE USE OF THE FORMULÆ.

Prove the following identities:

I. 1. $\tan\frac{A}{2} = \frac{\sin A}{1+\cos A}$.

2. $\cos 6A + \cos 2A = 2\cos 4A \cos 2A$.

3. $\frac{\sin A - \sin B}{\sin A + \sin B} = \frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)}$.

4. $2\operatorname{cosec} 2A = \sec A \operatorname{cosec} A$.

5. $\frac{\sin 6A - \sin 2A}{\sin 5A + \sin 3A} = 2\cot 4A \sin A$.

6. $\tan^2\frac{A}{2} = \frac{2\sin A - \sin 2A}{2\sin A + \sin 2A}$.

7. $\cos 3A = 4\cos^3 A - 3\cos A$.

8. $\operatorname{cosec} 2A + \cot 2A = \cot A$.

9. $\frac{\sin 6A - \sin 2A}{\sin 5A - \sin 3A} = 2\cos A$.

10. $\sin 3A = 3\sin A - 4\sin^3 A$.

II. 1. $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$.

2. $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$.

3. $2\cos 3A \sec A = \dfrac{\cos 4A + \cos 2A}{\cos^2 A}$.

4. $\cos A + \cos 3A = 2\cos 2A \cos A$.

5. $\dfrac{1}{1-\tan A} - \dfrac{1}{1+\tan A} = \tan 2A$.

6. $\cot A \tan 3A = \dfrac{\sin 4A + \sin 2A}{\sin 4A - \sin 2A}$.

7. $\sin 7A - \sin 3A = 2\sin 5A \cos 2A$.

8. $\sec 2A = \dfrac{\cot A + \tan A}{\cot A - \tan A}$.

9. $\dfrac{\sin A + \sin 3A + \sin 5A}{\sin 3A + \sin 5A + \sin 7A} = \dfrac{\sin 3A}{\sin 5A}$.

10. $\cos 2A + \cos 2B = 2\cos(A+B)\cos(A-B)$.

CHAPTER VIII.

TRIANGLES.

62 THE solution of triangles from certain given parts has been shortly noticed as being the chief use as well as the best practice in the use of the ratios.

The cases treated before have been the right-angled triangle, and the formula

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

There is a case in the use of the latter which may be noticed, as a good example of the discussion of the parts of a triangle.

63 *To solve a triangle having given two sides and the angle opposite to one of them.*

Let a, b be two sides, and A the angle opposite to one of them.

Then
$$\frac{\sin A}{a} = \frac{\sin B}{b},$$

$$\sin B = \frac{b \sin A}{a},$$

from which the value of $\sin B$ is given and B may be determined, except in certain cases.

1. If $b \sin A > a$, $\sin B > 1$, and so there can be no corresponding value of B, or there can be no triangle with the given parts.

2. If $b \sin A = a$, $\sin B = 1$, and B is a right angle.

3. If $b \sin A < a$, $\sin B < 1$, or the value of B is possible. But **an angle and its supplement have the same sine,** therefore there are two angles less than 180°, that is in a triangle, which are given by this known value of sin B, one greater and one less than a right angle.

i. If $a > b$, $A > B$, (Euc. I. 18) or B can only have the smaller value, for A and B cannot both be obtuse angles.

ii. If $a < b$, $A < B$, and there are two values of B, one the supplement of the other, and there are two triangles which have the given parts a, b, A.

This is called the **ambiguous case in the solution of triangles.**

These deductions may be exhibited geometrically.

64 *To construct the triangle from the given parts.*

The line AC is placed, so as to make the angle A with AB, and from it is cut off a part AC, equal to the line b, one of the given sides.

With centre C and distance a, the other side, a circle is described, which will cut AB in the other point B of the triangle.

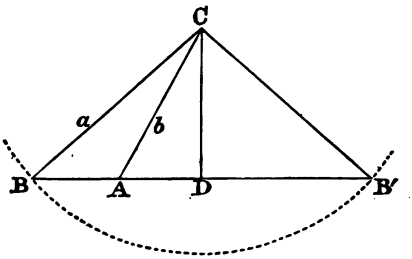
From C a perpendicular CD is drawn on AB.

$CD = b \sin A$.

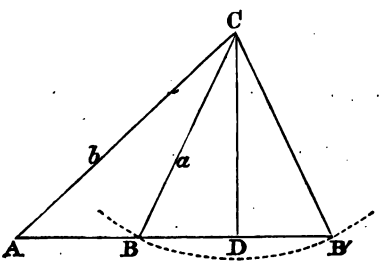
1. Now if $CD > a$ the circle will not cut AB, and there can be no such triangle.

2. If $CD = a$ the circle will touch AB, and there will be one triangle right-angled at B.

3. If $CD > a$, the circle will cut AB in two points B, B'.



i. If $a > b$, they will be on opposite sides of A, and the two triangles ACB, ACB' will not have the same angle A, but an angle and its supplement.



ii. If $a < b$, the two triangles ABC, AB'C will both have the angle A and the given sides a, b.

In this case the solution of the triangle from these parts is ambiguous.

65 *To find the length of the base of a triangle when two sides a, b, and the included angle* C *are given.*

We have found in § 29 that

$$a = b\cos C + c\cos B,$$
$$b = c\cos A + a\cos C,$$
$$c = a\cos B + b\cos A;$$

and therefore multiplying by a, b, c that

$$a^2 = ab\cos C + ac\cos B,$$
$$b^2 = bc\cos A + ab\cos C,$$
$$c^2 = ac\cos B + bc\cos A.$$

Adding the first two and subtracting the third,

$$a^2 + b^2 - c^2 = 2ab\cos C,$$

$$\therefore \mathbf{c^2 = a^2 + b^2 - 2ab\cos C},$$

with similar forms for a and b.

66 The parts of this triangle can be found in a **form more suitable to computation.**

$$\frac{\sin A}{\sin B} = \frac{a}{b}, \quad \therefore \frac{\sin A - \sin B}{\sin A + \sin B} = \frac{a-b}{a+b},$$

$$\therefore \frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)} = \frac{a-b}{a+b},$$

since $A + B = \pi - C$, *i.e.* $\tan\frac{1}{2}(A+B) = \cot\frac{C}{2}$;

$$\therefore \tan\frac{1}{2}(A-B) = \frac{a-b}{a+b}\operatorname{cotan}\frac{C}{2}.$$

Hence $A - B$ is found; and $A + B$ being known, the angles A, B, C and the sides a, b, c can be deduced.

67 *To solve a triangle when the three sides are given.*

We know that

$$a^2+b^2-c^2=2ab\cos C,$$

$$\therefore \cos C=\frac{a^2+b^2-c^2}{2ab},$$

$$1-\cos C=1-\frac{a^2+b^2-c^2}{2ab}=\frac{c^2-(a-b)^2}{2ab},$$

$$\sin^2\frac{C}{2}=\frac{(c-a+b)(c+a-b)}{4ab}.$$

Let $s=\frac{1}{2}(a+b+c)$, then $s-a=\frac{1}{2}(b+c-a)$ &c.

$$\therefore \sin^2\frac{C}{2}=\frac{(s-a)(s-b)}{ab}.$$

Also

$$1+\cos C=1+\frac{a^2+b^2-c^2}{2ab}=\frac{(a+b)^2-c^2}{2ab},$$

$$\therefore \cos^2\frac{C}{2}=\frac{(a+b+c)(a+b-c)}{4ab},$$

$$\therefore \cos^2\frac{C}{2}=\frac{s.(s-c)}{ab}.$$

Hence we have the values of the ratios of half the angles in terms of the sides;—

$$\sin\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{ab}},$$

$$\cos\frac{C}{2}=\sqrt{\frac{s.(s-c)}{ab}},$$

$$\tan\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{s.(s-c)}}.$$

68 In first dealing with triangles, acute-angled triangles alone were considered. **But the conventions there introduced have brought obtuse-angled triangles also into the same forms, so that the formulæ there found hold for all triangles.**

69 In collecting the formulæ for use it is suggested that in this form they may be useful as well as a reminder of the methods used, as of the facts themselves; they are

I. Formulæ connecting sides and angles of right-angled triangles; in twelve ways.

II. In any triangles

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

III. $a = b\cos C + c\cos B$, and two similar forms.

IV. $a^2 = b^2 + c^2 - 2bc\cos A$, „ „ „ „

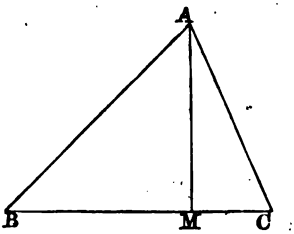
V. $\tan\frac{1}{2}(B-C) = \frac{b-c}{b+c}\cot\frac{A}{2}$ „ „ „

VI. $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$ „ „ „ „

$\cos\frac{A}{2} = \sqrt{\frac{s.(s-a)}{bc}}$ „ „ „ „

$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s.(s-a)}}$ „ „ „ „

70 *To find the area of a triangle.*



$$\text{Area} = \frac{1}{2}\text{BC}.\text{AM}$$

$$= \frac{1}{2}a \,.\, c \sin \text{B}$$

$$= ac \sin \frac{\text{B}}{2} \cos \frac{\text{B}}{2}$$

$$= ac \sqrt{\frac{s\,.\,(s-b)}{ac}} \sqrt{\frac{(s-a)\,(s-c)}{ac}}.$$

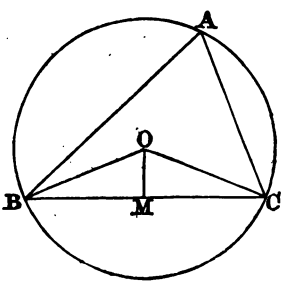
$$\text{Area} = \sqrt{s\,.\,(s-a)\,(s-b)\,.\,(s-c)}.$$

This is usually denoted by S, while the semi-sum of the sides $\frac{1}{2}(a+b+c)$ is denoted by s.

Hence $$\frac{\sin \text{A}}{a} = \frac{\sin \text{B}}{b} = \frac{\sin \text{C}}{c} = \frac{2\text{S}}{abc}.$$

71 *To find the radius of the circle which circumscribes a triangle.*

Joining the centre O to B and C, and drawing a perpendicular on BC.



Let the radius = R.

The angle BOC is twice the angle A,

so that $\text{BOM} = \text{A}$,

and $\text{R} = \text{BO}$

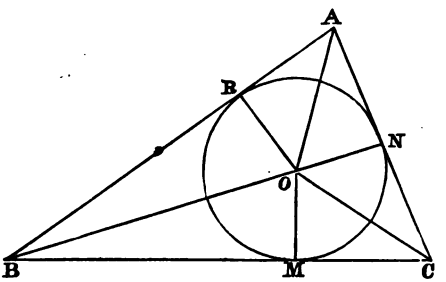
$= \text{BM} \operatorname{cosec} \text{BOM}$,

$$\therefore \text{R} = \frac{a}{2} \operatorname{cosec} \text{A} = \frac{a}{2 \sin \text{A}},$$

$$\text{R} = \frac{abc}{4s}.$$

72 *To find the radius of a circle inscribed in a triangle.*

Let the radius $= r$.



The area of the triangle is made up of the area of the three triangles AOB, AOC, COB.

Now the area of triangle AOC $= \frac{1}{2} b \,.\, r$;

$\therefore$ area of ABC $= \frac{1}{2} a \,.\, r + \frac{1}{2} b \,.\, r + \frac{1}{2} c \,.\, r$

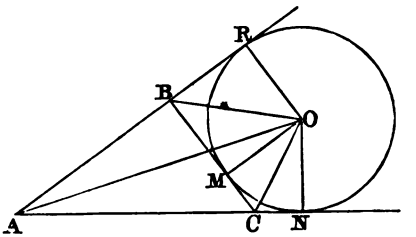
$= \frac{1}{2} (a + b + c) r$

$= s \,.\, r$;

so that $$r = \frac{\text{area of ABC}}{s} = \frac{S}{s}.$$

73 *To find the radius of the escribed circles.*

Let the radius of the circle escribed on the side a be r_1, and of those on the others r_2, r_3.



The area of ABC = AOB + AOC − BOC

$$= \frac{1}{2} c \,.\, r_1 + \frac{1}{2} b \,.\, r_1 - \frac{1}{2} a \,.\, r_1$$

$$= \frac{1}{2} (c + b - a)\, r_1$$

$$= (s - a)\, r_1;$$

$$r_1 = \frac{\text{area}}{s-a} = \frac{S}{s-a}; \text{ so } r_2 = \frac{S}{s-b}, \; r_3 = \frac{S}{s-c}.$$

CHAPTER IX.

LOGARITHMS.

74 THE calculations necessary to the finding of the parts of triangles, by the use of these formulæ, would be long and inconvenient, were it not for the use of logarithmic tables.

By the use of these the operations of mathematics are much simplified. **Multiplication of two numbers** is replaced by the **addition of their logarithms; Division** by the **subtraction of their logarithms;** while the tedious operation of finding roots and powers is effected by the **division and multiplication of the logarithms by the indices.**

A logarithm is an index. If $a^x = m$, x is called the logarithm of m to the base a, and it will be seen that the operations spoken of are just those which take place with the indices of numbers.

75 The best definition of a logarithm that can be given is $a^{\log_a x} = x$, when $\log_a x$ is read "the logarithm of x to the base a."

Thus $10^{\cdot 301030} = 2$. Here ·301030 is the logarithm of 2 to the base 10, the index of the power to which 10 is raised.

Using this definition we have

$$x = a^{\log_a x},$$

$$y = a^{\log_a y}.$$

Multiplying, $a^{\log_a xy} = xy = a^{\log_a x + \log_a y}.$

$$\therefore \log_a xy = \log_a x + \log_a y. \quad (1)$$

Dividing, $a^{\log_a \frac{x}{y}} = \frac{x}{y} = a^{\log_a x - \log_a y}.$

$$\therefore \log_a \frac{x}{y} = \log_a x - \log_a y. \quad (2)$$

Raising to the power m, $a^{\log_a x^m} = x^m = a^{m \log_a x}.$

$$\therefore \log_a x^m = m \log_a x. \quad (3)$$

76 1. The logarithm of a product is equal to the sum of the logarithms of the factors, as

$$\log_{10} 6 = \log_{10} 3 + \log_{10} 2.$$

2. The logarithm of a quotient is equal to the difference of the logarithms of the factors, as

$$\log_{10} 5 = \log_{10} 10 - \log_{10} 2.$$

3. The logarithm of a power is equal to the product of the index of the power, and the logarithm of the number, as

$$\log_{10} 3^2 = 2 \log_{10} 3.$$

77 Tables are computed which give the logarithms of numbers, and the logarithms of the sine and cosine of angles less than 45°.

All the preceding formulæ may be written in the form adapted for the use of tables, if they are in the form of products.

The logarithm of a sum is not the sum of logarithms, so that for this method of computation those which involve sums only are useless, *e.g.*

$$a = b\cos c + c\cos \mathrm{B}.$$

Examples of the adaptation of formulæ:

(1) $a = c\sin \mathrm{A}$, § 27.

$$\log a = \log c + \log \sin \mathrm{A}.$$

(2) $\sin\frac{\mathrm{A}}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$, § 67.

$$\log\sin\frac{\mathrm{A}}{2} = \frac{1}{2}\{\log(s-b) + \log(s-c) - \log b - \log c\}.$$

(3) $\tan\frac{1}{2}(\mathrm{B}-\mathrm{C}) = \frac{b-c}{b+c}\operatorname{cotan}\frac{\mathrm{A}}{2}$, § 66.

$$\log\tan\frac{1}{2}(\mathrm{B}-\mathrm{C}) = \log(b-c) - \log(b+c) + \log\cot\frac{\mathrm{A}}{2}.$$

78 In referring to a table of logarithms, numbers will be found in this form:

N	0	1	2
825	916454	6507	6559

, and so on for 3, 4, &c.

This means that

$$\log 8250 = 3{\cdot}916454,$$
$$\log 8251 = 3{\cdot}916507,$$
$$\log 8252 = 3{\cdot}916559.$$

The numbers given in the table are decimals and are the logarithms of 8·250, 8·251, and so on, for numbers which lie between 0 and 10 must have logarithms between 0 and 1, that is, pure decimals.

A number is placed before these decimals which depends on the **value of the place of the highest digit.** This is called the characteristic, since it denotes the integral part of the power of 10.

If this place be 10^n, for instance millions, the characteristic is n, in that case 6,

$$\log 8252000 = 6{\cdot}3916559.$$

If in a decimal the highest place be 10^{-n}, the characteristic is $-n$; tens of thousandths giving $\bar{4}$.

$$\log {\cdot}0008251 = \bar{4}{\cdot}3916507.$$

79 The theory of logarithms simply is that which has been given here, that a logarithm follows the rules of indices; but it will be understood that the use of logarithms can only be mastered by practice.

Those who wish to follow out the practical part of the subject will now take up books on Mensuration. Prof. Elliott's books are good instances of the kind.

Those who wish to attend to the theoretical questions of Trigonometry will find that what has been said here will be a stepping stone to Todhunter's larger Trigonometry.

MISCELLANEOUS EXAMPLES.

I. 1. SHEW that neither the sine nor the cosine of an angle can be greater than unity.

2. Prove that

$$\cot^2 A + \operatorname{cosec}^2 A = (\sin^2 A + 2\cos^2 A)\operatorname{cosec}^2 A.$$

3. Find $\sin 25^\circ$ and $\cos 15^\circ$ from the ratios of $\frac{\pi}{4}$ and $\frac{\pi}{6}$.

4. At what height will a bullet hit a house, when it is fired from the ground 20 yards off, at an elevation of 15°?

5. What are the measures of the angle of a quindecagon?

6. If one side of a triangle be 15 ft. and the adjacent angles be 60° and 75°; find the other angle and the sides.

7. What is the angular elevation of the sun when a vertical stick casts a shadow as long as itself?

II. 1. I stand in a line with one side of a rectangular building at a distance of 35 yds. from the corner, and the house subtends an angle of 30° at my eye. How long is that side which I see?

2. A triangle has sides $a = 3$, $b = 2$. Find between what limits $\sin B$ must lie that the triangle may be possible and ambiguous. Must the triangle be ambiguous if it is possible?

3. Prove that $\sin\phi \cdot \tan\phi = \sec\phi - \cos\phi$.

4. Assuming $\cos A = \frac{1}{2}$; what different angles are denoted?

5. I go 3 miles up a slope whose cosine is $\frac{4}{9}$, and 5 miles down one whose cosine is $\frac{4}{5}$, what is the whole distance gone on level ground?

6. Find the French measure of $\frac{\pi}{5}$; $\frac{3\pi}{4}$ and 81°.

7. Prove that $\frac{\sin 8\theta - \sin 2\theta}{\sin 7\theta - \sin\theta} = \cos 5\theta \sec 4\theta$.

III. 1. Alter to French measure (*metre* = 39 *in.*) the following statement. A field has two sides of 300 yds. and 200 yds. inclined to one another at an angle of 89°.15′, while one of the other sides of 150 yds. is inclined at an angle of 105°.2′ to the latter. Write, in a form adapted to logarithms, the other side and the area of the field.

2. A gable of a house 30 ft. wide is 10 ft. high; find some measure of its angles.

3. Prove that $\sec A + \sec 3A = 2 \cos 2A \,.\, \sec 3A$.

4. Find the radius of a circle escribed on the greatest side of a triangle whose sides are 5; 4; 3.

5. Find the side a of a triangle when the other sides are 3 and 4, and $\cos B = \cdot 3$, and $\cos C = \cdot 4$.

6. What is the length of a tire of a wheel whose radius is 52 in.? What part of it is cut off by two spokes including an angle 1·5?

7. Prove that

$$\sin\theta \tan\theta + \cos\theta \cot\theta$$

$$= \frac{1}{2}(\sec\theta + \operatorname{cosec}\theta)\,(2 - \sin 2\theta).$$

IV. 1. Find the degrees, minutes and seconds in the angles $34^g.15^c.21^{cc}$; $121^g.8^c.31^{cc}$; $\frac{\pi}{15}$; $\frac{3\pi}{11}$; 1·003.

2. Find the diagonals of a parallelogram, of angle 60° included by the sides 5 and 3.

3. What is the area of the triangle whose

$$A = 35°.16'; \; b = 35{\cdot}71 \text{ ft.}; \; c = 16{\cdot}21 \text{ ft.}?$$

Given

$$\log \sin 35°.16' = 9{\cdot}7614638 - 10$$
$$\log \quad 3571 \quad = 3{\cdot}5527899$$
$$\log \quad 1621 \quad = 3{\cdot}2097830$$
$$\log \quad 33422 \quad = 4{\cdot}5240324$$
$$\log \quad 33423 \quad = 4{\cdot}5240454.$$

4. A railway cutting is 12 yds. deep, 30 yds. broad at the top, and its slopes are 15 yds. long. Find the breadth of the railway and the ratios of the angle of the slopes to the horizon.

5. Prove that $\cot\phi + \operatorname{cosec}\phi = \cot\frac{\phi}{2}$.

6. A hurdle is composed of three bars 7 ft. long, two uprights at their ends 3·5 ft. long, and two cross bars joining the corners. Find the angle made by the latter with all the bars.

7. Given $\tan A = 21$, $\tan B = 20$, in a triangle, find $\tan C$.

V. 1. Alter the expressions $\cos\phi = \frac{\sqrt{3}}{2}$; arc 40 ft., rad. 30 ft.; $\theta = \frac{\pi}{5}$ or $\frac{2\pi}{3}$; $A = 35^\circ$ or 18° to French measure.

2. Two ships leave harbour; one sails N.N.E. at a rate of 8 knots an hour, the other sails northwards and keeps the first always due East of her. How fast does the latter sail?

3. In a triangle, $A = 35^\circ.15'$, $B = 82^\circ.48'$, $a = 15{\cdot}2$ ft., $b = 1048{\cdot}2$ in. Find c, given

$$\cos 35^\circ.15' = {\cdot}8166\,;\ \sin 7^\circ.12' = {\cdot}1253.$$

4. Prove that

$$\operatorname{cosec}(A + C) = \operatorname{cosec} B,$$

and

$$2(\sin A + \sin B)(\cos A + \cos B) = \sin 2A + \sin 2B + 2\sin C,$$

when A, B, C are angles of a triangle.

5. A river is 350 ft. wide, and a man swims to the other side in 700 ft. How much is he carried down stream, and at what angle to the bank?

6. Prove that

$$\sec 75^\circ - \sec 15^\circ = 2\sqrt{2}.$$

7. Find the radius of the circle inscribed in a triangle whose sides are 5; 11; 21.

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